Monday 10/21 Lecture Notes

Thm Let A be an n×m matrix and define $T(\vec{x}) = A\vec{x}$. This is a linear transformation. (THIS WILL BE ON FINAL!!!)

Proof: Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{R}^m$. $A = [a_1, \dots, a_m]$, $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$ then

Q: Can you show something isn't an LT?

$T(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = [\vec{a}_1, \dots, \vec{a}_m]\begin{bmatrix} u_1+v_1 \\ \vdots \\ u_m+v_m \end{bmatrix} = [(u_1+v_1)a_1 + \dots + (u_m+v_m)\vec{a}_m$

$= u_1a_1 + \dots + u_ma_m + v_1a_1 + \dots + v_ma_m = A\vec{u} + A\vec{v} = T(\vec{u}) + T(\vec{v})$

* $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

* $A(c\vec{u}) = c(A\vec{u})$

Ex $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 3x_2 \\ x_2 + 4x_3 \end{bmatrix}$ is a linear transformation, because we can rewrite it as a matrix multiplication:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Range ⇒ is $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ in the range? ⇒ $T\left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ ⇒ set $= \begin{smallmatrix} 2 \\ 4 \end{smallmatrix}$ ⇒ $\left[\begin{array}{ccc|c} 1 & -3 & 0 & 2 \\ 0 & 1 & 4 & 4 \end{array}\right]$

$x_3 = s_1,\ x_2 = 4 - s_1,\ x_1 = 14 - 12s_1$

So, any vector in the form: $\overset{14}{x_1} = \begin{bmatrix} 14 \\ 4 \\ 0 \end{bmatrix} + s_1\begin{bmatrix} -12 \\ -4 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Theorem: Let $A = [a_1, \dots, a_m]$ be a n×m matrix and let $T: \mathbb{R}^m \to \mathbb{R}^n$ w/ $T(\vec{x}) = A\vec{x}$ then:

same # columns as entries x

same # rows as entries

a) $\vec{v}$ is in the range of T if $A\vec{x} = \vec{v}$ has solutions

b) range of $T = \text{span}(\vec{a}_1, \dots, \vec{a}_m)$

Ex $T(x) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}\vec{x}$ is range $(T) = \mathbb{R}^3$?

No! Because 2 vectors cannot span $\mathbb{R}^3$. Range $(T) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}\right) \neq \mathbb{R}^3$

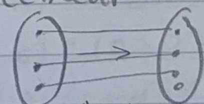
Definition: Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation

(surjective) a) T is called one to one (1-1) if for every vector $\vec{v}$ in $\mathbb{R}^n$ there is at most one vector $\vec{u}$ in $\mathbb{R}^m$ such that $T(\vec{u}) = \vec{v}$

(injective) b) T is called onto if for every vector $\vec{v}$ in $\mathbb{R}^n$ there is at least one vector $\vec{u}$ in $\mathbb{R}^m$ such that $T(\vec{u}) = \vec{v}$

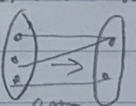
Alternatively: T is 1-1 if $T(\vec{u}) = T(\vec{v})$ implies $\vec{u} = \vec{v}$

(linear transformations do not need to be 1-1 or onto)



1-1

everything gets hit at most once maybe not everything gets hit



onto

every pt gets hit, maybe more than once

Theorem: Let T be a LT. T is 1-1 if and only if $T(\vec{x}) = \vec{0}$ has the solution $\vec{x} = \vec{0}$

Pf: know $T(\vec{0}) = \vec{0}$ (⇒) if T is 1-1 since $T \neq \vec{0}$ then $T(\vec{x}) = \vec{0}$ means $\vec{x} = \vec{0}$

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Suppose $T(\vec{x})=\vec{0}$ has only the solution $x=0$, consider $T(u)=T(v)$ then $T(\vec{u})-T(\vec{v})=\vec{0}$ so $T(\vec{u}-\vec{v})=0$ thus $\vec{u}-\vec{v}=\vec{0}$ and $\vec{u}=\vec{v}$ and T is 1-1.

Theorem: Let A be a $n \times M$ Matrix, and $T(\vec{x})=A\vec{x}$ then

(a) T is onto if and only if span of columns of $A=\mathbb{R}^n$

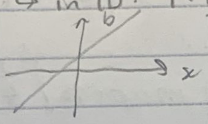
(b) T is 1-1 if and only if columns are linearly independent

Theorem: Let T be a linear transformation

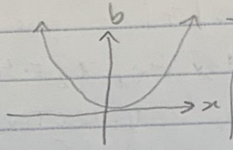
Onto and 1-1 Review

A transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector b in $\mathbb{R}^m$ is the image of some x in $\mathbb{R}^n$

↳ in 1D: $T:\mathbb{R}\to\mathbb{R}$



for ~~any value of~~ x, you can get to any value of y (b).



Since you can't get below 0, not onto.

A transformation is onto if for $T:\mathbb{R}^n \to \mathbb{R}^m$, $[A \ \vec{b}]$ is consistent for all $\vec{b}$ in $\mathbb{R}^m$.

Ex if $\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Reduced A}}$ then $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$ is inconsistent bc of last row. So not onto

Ex

$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ then $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & \overset{b}{\downarrow} 1 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$

Without even augmenting a solution, we can see its onto because theres a pivot in every row in echelon form